HOW TO BUILD YOUR LATENT MARKOV MODEL: THE ROLE OF TIME AND SPACE

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ABSTRACT: In empirical research involving latent Markov models, there is a tendency of research communities building up expertise on one particular class of such models, then shoehorning any given data set into that very model formulation. This talk attempts to overcome this myopia by offering a unifying view on what otherwise are often considered completely separate model classes — from hidden Markov models to Cox processes — thereby providing guidance as to how a latent Markov model formulation can be suitably tailored to the data at hand.

KEYWORDS: Cox process, hidden Markov model, state-space model.

1 Introduction

Over the last two decades, latent Markov models* have taken applied research by storm. This success story can be explained by their intuitive appeal, their mathematical tractability, and the various types of inference they allow for. Yet, while empirical researchers are well-acquainted with the various flavours of regression, the same cannot be said for latent Markov models. Instead, a tendency can be recognised that researchers focus on building expertise on one particular type of such models, and then shoehorn any given data set into the model they happen to know best.

The challenge of identifying a suitable model formulation for a given data set primarily concerns two choices to be made: whether to use a discrete-time or a continuous-time model formulation, and whether to assume a discrete or a continuous state space. Classifying different model classes along these two dimensions, we provide an overview of the most relevant classes of latent

*i.e. stochastic process models for sequential data driven by latent Markovian processes; note these may also be referred to as, *inter alia*, state-space models, hidden Markov models, doubly stochastic processes, or dependent mixture models — we use the label "latent Markov model" as it appears to be a good umbrella term for all special cases considered in this paper

Markov models and emphasise that the inferential methods for these different classes are for the most part effectively identical, such that there is no reason why researchers should focus on any one class of these models.

2 Overview of latent Markov model formulations

Table 1 attempts to classify the main types of latent Markov models according to the type of states, either discrete or continuous, the observed process is assumed to be driven by, and the role of time, i.e. the mathematical operationalisation of the times at which the sequential observations are made.

Table 1. *Classification of six popular classes of latent Markov models according to the respective role of time and space.*

	discrete states	continuous states
discrete time	*(A)* (basic) hidden Markov model	*(B)* (basic) state-space model
continuous time, non-inform. obs. times	*(C)* continuous-time hidden Markov model	*(D)* continuous-time state-space model
continuous time, inform. obs. times	*(E)* Markov-modulated Poisson process	*(F)* Cox process

The simplest case *(A)* arises when the states are discrete and the process is modelled in discrete time, i.e. as a time series $\{X_t\}_{t=1,\dots,T}$. In its basic dependence structure, the corresponding hidden Markov model (HMM) is defined by an N–state homogeneous Markov chain $\{S_t\}_{t=1,\dots,T}$ as the state process, specified by the initial distribution $\delta = \big(\delta_1,\dots,\delta_N\big)$, $\delta_i = \Pr(S_1 = i)$, and the $N \times N$ transition probability matrix $\Gamma = (\gamma_{ij})$, as well as the N emission distributions $f_1(x_t),\dots,f_N(x_t)$, which are selected by the state process. An intuitive example is animal movement, where the observations $x_1,\dots,x_T$ could be the hourly step lengths of an animal and the states the behavioural modes (cf. Beumer *et al.*, 2020). HMMs are mathematically tractable, as recursive techniques can be used for likelihood evaluation, state decoding, and forecasting.

In many settings, it will however not be reasonable to assume that the state process $\{S_t\}$ is discrete-valued. For example, the volatility underlying share returns evolves gradually over time. In such cases, it is more adequate to model the discrete-time state process $\{S_t\}_{t=1,\dots,T}$ as an autoregressive process,

$$s_t = \phi(s_{t-1} - \mu) + \mu + \sigma\varepsilon_t, \quad \varepsilon_t \overset{iid}{\sim} N(0,1),$$

with long-term mean $\mu \in \mathbb{R}$, persistence parameter $-1 < \phi < 1$ and standard deviation $\sigma > 0$ of the error process, and with the distribution of x_t in some way depending on s_t. Such a model is commonly referred to as *(B)* state-space model (SSM), and there are many different techniques for estimating the associated parameters, ranging from the Kalman filter to MCMC-based methods (Auger-Méthé *et al.*, 2021). While this plethora of estimation techniques can be intimidating for practitioners, it is worth pointing out that SSMs can conveniently be approximated using HMMs with a large state space.

Basic HMMs or SSMs both need to be modified when the intervals between observation times are not of the same length. Such temporally irregular sampling schemes are quite common for example in medical or survey data. If in such cases a discrete state space seems adequate, then $\{S_t\}_{0\leq t\leq T}$ can be modelled as a continuous-time Markov chain, specified by the infinitesimal generator matrix $\mathbf{Q} = (q_{ij})$, with state transition intensities

$$q_{ij} = \lim_{\Delta t \to 0} \frac{\Pr(s_{t+\Delta t} = j \mid s_t = i)}{\Delta t},$$

leading to *(C)* a continuous-time HMM. If instead the states ought to be modelled as continuous-valued, then a stochastic differential equation (SDE) can be used, e.g. the Ornstein-Uhlenbeck process

$$ds_t = \theta(\mu - s_t)dt + \sigma dw_t,$$

where w_t is the Brownian motion and $\theta > 0$ controls the strength of reversion to the long-term mean μ. Such a model would most naturally be labelled *(D)* a continuous-time SSM. For inference, recursive techniques similar to the discrete-time case are available (Jackson *et al.*, 2003; Mews *et al.*, 2022b).

Finally, we need to distinguish cases where the observation times themselves are informative, e.g. in medicine, when longitudinal observations are made whenever a patient goes to a doctor, likely indicating sickness. In such cases, *(E)* Markov-modulated Poisson processes (MMPPs) can be used to model a system traversing through a finite state space, with the observation times modelled as a Poisson arrival process with rate λ_{s_t} depending on the state s_t currently active. Such a model can be further extended by including marks, say for modelling biomarkers measured at each consultation (Mews *et al.*, 2022a).

If assuming only finitely many states of such a process is inadequate, then the continuous-time Markov chain model for $\{S_t\}$ can again be replaced by an SDE, leading to the class of *(F)* Cox processes.

3 Conclusion

By classifying latent Markov models according to the assumptions made concerning time and (state) space, we promote a more unified view on what otherwise are often considered fairly separate model classes. This categorisation is far from perfect — for example, as it stands it does not have a place for SDEs driven by latent states — however, we hope that it can provide some guidance for empirical researchers when making their modelling decisions. The main point we are trying to make is that "you should model the process that gives rise to the data, not shoehorn the data into a model you happen to have at hand" (quote by David L. Borchers, pers. communication) — and to be able to do the former, it is important to have a big picture view of the model classes available.

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